Brief Communication

On the Inadequacy of Hertzian Solution of Two-dimensional Line Contact Problems

Krishna P. Singh
President and CEO
Holtec International
Marlton, NJ

I. Introduction

Hertz's solution yields an infinite approach corresponding to a finite load for the case of two long cylinders in line contact. This is a physically meaningless solution. This note is addressed to this deficiency in Hertz's solution. An expression for the approach and shape of the deformed surface is derived. This result may prove to be of great value in determining the compliance of long cylindrical elements, e.g. needle bearings, etc.

Notation

- b semi-width of contact strip
- E Young's modulus
- F normal thrust
- R radius of cylinder
- R' radius of imaginary rigid cylinder
- δ rigid body approach
- ν Poisson's ratio

The contact of two long cylinders of dissimilar materials and different radii along their generators was considered by Hertz in 1882 (**1**). Hertz treated this problem as a limiting case of the three-dimensional problem of contact of two quadratic non-conformal surfaces. Although Hertz's prescription gives the width of the contact strip and the pressure distribution accurately, the compliance of the two bodies cannot be determined. Hertz recognized the limitation of his solution and pointed out that the local curvature of the contacting surfaces must be considered in determining the influence function (Green's function to correlate surface displacement to surface traction). This inadequacy in the Hertzian solution has been a major impediment in determining the deflection of machine elements, e.g. needle bearings [see Harris (**2**)].

A general analytical expression for compliance is constructed in this article which removes the aforementioned deficiency in the Hertzian solution. In what follows, subscripts 1 and 2 are ascribed to the quantities pertaining to cylinders 1 and 2, respectively.

According to Hertz, the semi-width b of the contact strip is given as a function of the normal thrust F (per unit length), and elastic constant of the two bodies, as follows:

$$b = \left(\frac{4F\{[(1-\nu_1^2)/\pi E_1]+[(1-\nu_2^2)/\pi E_2]\}}{R_1^{-1}+R_2^{-1}}\right)^{0\cdot5}, \tag{1}$$

where ν, E and R with appropriate subscripts denote the Poisson ratio, Young's modulus and radius of the cylinders. Dörr (**3**) showed that if a cylinder of radius R is compressed between rigid planes, and the approach, δ, of the two planes is given by

$$\delta = \frac{4F(1-\nu^2)}{\pi E}\left(\ln\frac{4R}{b}-\frac{1}{2}\right). \tag{2}$$

When two cylinders of dissimilar materials and different radii are pressed together, the segment of the arc in contact will, in general, flatten out, and the local radii of curvature of the deformed cylinders, R', will vary over the width of contact. However, it is intuitively obvious that R' at all points in the deformed segment would be considerably larger than the characteristic dimension of the contact strip, $2b$. The contact of cylinders of radii R_1 and R_2 may be broken into contact of a cylinder 1 of radius R_1 with a rigid cylinder of radius R', and contact of the cylinder of radius R_2 with a rigid cylinder of radius $-R'$. Noting that $R' \gg 2b$, the rigid cylinder approximates a rigid flat plate, and Dörr's prescription may be invoked. It is also recognized that in determining the diametral compression of the elastic cylinder by two rigid planes, Dörr takes into consideration the curvature effect of the elastic cylinder. Assuming one of the rigid planes to be stationary, the movement of the center line of the elastic cylinder is given by $0\cdot5\delta$, where δ is given by Eq. (2). Thus, displacement of the center line δ_1 of cylinder 1 is given as

$$\delta_1 = \frac{2F(1-\nu_1^2)}{\pi E_1}\left(\ln\frac{4R_1}{b}-\frac{1}{2}\right), \tag{3}$$

where b, the semi-width of contact region, is given by Hertz's solution by setting $E_2 = \infty$ and $R_2 = R'$ in Eq. (1). Thus,

$$b = \left(\frac{4F(1-\nu_1^2)}{\pi E_1(R_1^{-1}+R'^{-1})}\right)^{\frac{1}{2}}. \tag{4}$$

Similar reasoning for the contact of cylinder 2 with the rigid cylinder of radius $-R'$, yields

$$\delta_2 = \frac{2F(1-\nu_2^2)}{\pi E_2}\left(\ln\frac{4R_2}{b}-\frac{1}{2}\right) \tag{5}$$

and

$$b = \left(\frac{4F(1-\nu_2^2)}{\pi E_2(R_2^{-1}-R'^{-1})}\right)^{\frac{1}{2}}. \tag{6}$$

An expression for R' is extracted from Eqs. (4) and (6), which is

$$R' = \frac{1+\alpha}{\alpha R_2^{-1} - R_1^{-1}}, \tag{7}$$

where

$$\alpha = \frac{1-\nu_1^2}{1-\nu_2^2}\frac{E_2}{E_1}. \tag{8}$$

Let $\rho = R_1/R_2$, $\rho' = R'/R_1$, then

$$\rho' = \frac{1+\alpha}{\alpha\rho - 1}. \tag{9}$$

Furthermore, substituting in Eq. (4) for R' from Eq. (7) recovers Eq. (1), which confirms the validity of these results. The total approach, δ, of the two cylinders is given from Eqs. (3) and (5).

$$\delta = \delta_1 + \delta_2 \tag{10a}$$

or

$$\delta = 2F\left[\frac{1-\nu_1^2}{\pi E_1}\left(\ln\frac{4R_1}{b} - \frac{1}{2}\right) + \frac{1-\nu_2^2}{\pi E_2}\left(\ln\frac{4R_2}{b} - \frac{1}{2}\right)\right]. \tag{10b}$$

Equation (10b) furnishes the compliance of the contacting cylinders.

Notice ρ' and ρ are readily calculated from Eq. (9), and thus the approximate curvature of the deformed surface is readily determined. It is of some interest to note that the approach δ is a transcendental function of the thrust F or the characteristic length of contact region, $2b$. In the three-dimensional contact of quadratic surfaces (Hertzian solution) δ is proportional to $F^{\frac{2}{3}}$ and is proportional to the square of the characteristic dimension of the contact region (elliptical for Hertzian contact problems).

In the event that the contacting cylinders have a small length-to-radius ratio, the two-dimensional assumption may not be valid, and the prescription suggested in this article may be in error. A more general numerical solution for the line contact problems has been recently given by Singh and Paul (**4**).

References

(**1**) H. Hertz, "Miscellaneous Papers", translated by D. E. Jones and G. A. Schott, Macmillan, London, 1896. (Translation from "On the contact of rigid elastic solids and on hardness", *Verhandlungen des Vereins zur Beförderung des Gewerbefleisses*, Nov. 1882.)

(**2**) T. A. Harris, "Rolling Bearing Analysis", Wiley, New York, 1966.

(**3**) J. Dörr, "Oberflächenverformungen und Randkräfte bei runden Rollen und Bohrungen", *Stahlbau*, Vol. 24, pp. 202–206, 1955.

(**4**) K. P. Singh and B. Paul, "Stress concentration in crowned rollers", *J. Engnr. Industry, Trans. ASME*, 1974, to be published.