Reprinted from October 1982, Vol. 104, Journal of Engineering for Power

K.P. Singh
President and CEO
Holtec International
Marlton, NJ

An Approximate Method for Evaluating the Temperature Field in Tubesheet Ligaments of Tubular Heat Exchangers Under Steady-State Conditions

A method to determine the temperature field in a typical tubesheet ligament is developed. Using a numerical example, it is shown that, contrary to the common belief, a temperature gradient may exist throughout the thickness of the tubesheet in heat exchangers operating under "typical" conditions. Such thermal gradients can have significant influence on the exchanger's thermal and mechanical performance. The solution is developed with an eye to its suitability as a design tool.

Introduction

Tubesheets in shell-and-tube heat exchangers form a principal barrier between two pressure chambers, commonly referred to as "shell-side" and "tubeside" in the literature. The two chambers usually operate at different temperatures resulting in a thermal gradient in the body of the tubesheet. Gardner [1] showed that the ligament material of the tubesheet is mainly at the tubeside fluid temperature except for a thin skin facing the shellside fluid where a steep temperature gradient exists. Therefore, the so-called "thermal skin effect" produces only localized stresses (known as "peak stresses" in the lexicon of the ASME Codes). The body of the tubesheet does not experience a "global" thermal stress.

The intent of this paper is to show that under certain circumstances the thermal gradient may not be limited to the shellside skin. In fact, the thermal gradient may introduce significant rotation at the tubesheet rim resulting in leakage where the tubesheet is also used as a flange. Such a situation occurs whenever the shellside fluid heat-transfer coefficient is high, and the tube material possesses low thermal conductivity.

The knowledge of tubesheet/tubewall temperature field is especially important in the case of "rolled only" joints. The interfacial pressure between the tubewall and tubesheet provides the protection against leakage between the two pressurized chambers. Under the operating condition, the development of a differential temperature in conjunction with different coefficient of the wall expansion between the tubesheet ligament and tubewall may reduce the interface pressure below the threshold necessary to maintain a leak proof joint. Industrial design standards [2] provide empirical guidelines on the maximum allowable "joint temperature." A detailed knowledge of the temperature distribution in the tubewall and tubesheet ligament region is the first logical step towards the formulation of a mathematical criterion for determining the sealworthiness of "rolled only" joints.

A similar situation arises in the case of double tubesheet designs. The tubesheet in contact with the shellside fluid may be at a substantially different temperature than the tubeside tubesheet if the shellside heat transfer is much higher than the tubeside coefficient. In such an event, the tube segments spanning the two tubesheets may experience severe bending and shear stresses [3].

Gardner's Equation for tubesheet ligament temperature, despite its simplicity, is fairly accurate for commercial heat exchangers. However, as heat exchangers are deployed in ever more exacting situations, the need to formulate a solution capable of predicting the temperature field with accuracy becomes imperative. This paper is an attempt to fulfill this need.

While we seek to devise a solution of maximum generality, the need to keep the associated computation effort reasonably small is equally important. It is important that the method should lead to a solution procedure which can be utilized in a programmable calculator or a small computer. Availability of such a solution will most definitely encourage a check of tubesheet ligament temperatures during routine design work.

The solution described in this paper meets the foregoing criteria of simplicity and generality. Since the solution implies many approximations of heat-transfer equations, a thorough comparison with "Finite Element" solutions was performed to establish its accuracy and reliability. A typical comparison is described in this paper.

Due to the rather obscure source of Gardner's paper, an abstract of his solution is given in Appendix A for reference purposes. A numerical example illustrates a comparison between Gardner's formula and the solution constructed in this paper.

Contributed by the Nuclear Engineering Division and presented at the ASME Winter Annual Meeting, Washington, D.C., November 15-20, 1981. Manuscript received by the Nuclear Engineering Division November 19, 1981. Paper No. 81-WA/NE-12.

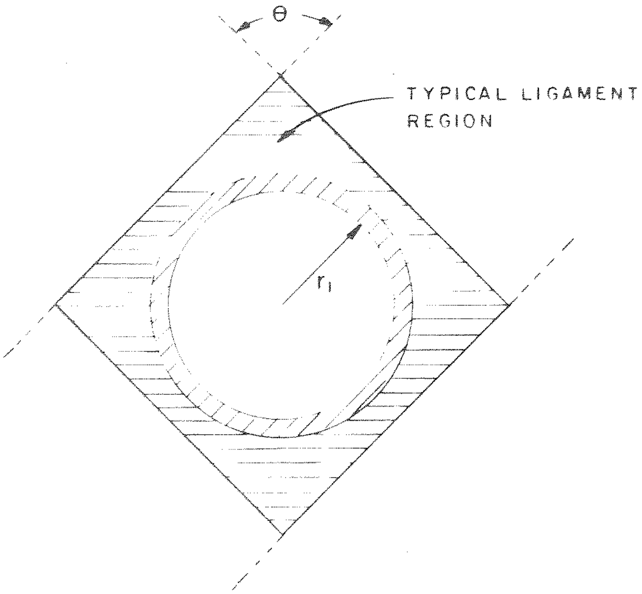

Fig. 1 Typical tube and its associated ligament cross section

In the following section a brief description of the mathematical model and its analysis is provided. Design engineers interested in using the final equations without the burdensome details of derivation may proceed directly to section 3 wherein a step-by-step description of the calculation procedure is given. Numerical illustration of the method and its comparison with a Finite Element Code solution is provided in section 4. Concluding remarks are contained in section 5.

Analysis

The cross section of a typical ligament surrounding a tube is shown in Fig. 1. During steady-state conditions, the temperature field in all ligaments will be identical, hence we will focus our attention on a composite region consisting of one tube, and its associated tubesheet ligament. It is also known that the variation of temperature is mainly in the direction of the tube axis.

In other words, the temperature changes in the lateral (in-plane) direction are, relatively speaking, quite small. These considerations lead us to model the tube-ligament assemblage as two interfacing coaxial cylinders. The inner cylinder is the tube which contains a fluid at temperature T_t. The outer cylinder is the tubesheet ligament. The noncircular cross section of the ligament is replaced by an equivalent circular section of equal cross-sectional area. This approximation is tenable so long as the temperature gradient in the lateral direction is small. It can be shown that the outer radius of the equivalent cylinder is given by

$$r_3 = p\left(\frac{\text{Sin}\,\theta}{\pi}\right)^{1/2} \tag{1}$$

where θ is the layout angle (Fig. 1), and p is the layout pitch. Thus, the idealized problem under consideration is that of two coaxial cylinders subject to a heat source at tempertures T_t on the inside cylindrical surface, T_s on the outside exposed surface of the inner cylinder and right lateral surface of outer cylinder, and T_t on the left lateral surface of both cylinders (Fig. 2). The inner cylinder is assumed to be infinitely long. For the purpose of analysis, the inner cylinder is subdivided into two regions, indicated by 1 and 1′ in Fig. 2. Assuming regions 1 and 2 to be at temperature T_1 and T_2 (functions of x) respectively, the governing heat-conduction equations are:

$$k_1\frac{d^2T_1}{dx^2} + \frac{[h_tS_1(T_t-T_1)-q_{12}S_2]}{a_1} = 0 \tag{2}$$

where

S_1 = circumference of inner tube surface
S_2 = circumference of tube outer surface
q_{12} = heat flux from region 1 to region 2
k_1 = conductivity of region 1 (tube) material
a_1 = cross-sectional area of region 1

Note: $S_1 = 2\pi r_1;\ S_2 = 2\pi r_2$ (3a)

$$a_1 = \pi(r_2^2 - r_1^2) \tag{3b}$$

Similarly, the governing equation for region 2 is

$$k_2\frac{d^2T_2}{dx^2} + \frac{q_{12}S_2}{a_2} = 0 \tag{4a}$$

where $a_2 = \pi(r_3^2 - r_2^2)$ (4b)

Let $q_{12} = h_{12}(T_1 - T_2)$ (5)

where h_{12} is the equivalent inter-region heat-transfer coefficient. An approximate expression for h_{12} is derived later

Nomenclature

$[A]$ = coefficient matrix (4×4)
a_1 = lateral cross-sectional area of region 1 (Fig. 2)
a_2 = lateral cross-sectional area of region 2 (Fig. 2)
b_i = constants of integration, $i=1,2,3,4$ (equation 11)
c_i = derived coefficients, $i=1,2,3,4$ (equation 10)
d_i = constants of integration, $i=1,2,3,4$ (equation 13)
$\{f\}$ = right hand side vector (4 elements)
h_{12} = heat-transfer coefficient corresponding to q_{12} (equation 5)
h_c = heat-transfer coefficient on the tubesheet surface in the tubeside chamber
h_t = in-tube heat-transfer coefficient
h_s = shell-side heat-transfer coefficient on the tube-sheet surface
h_s' = shellside heat-transfer coefficient on tube outer surface
k_1 = thermal conductivity of tube material
k_2 = thermal conductivity of tubesheet material
m_1, m_2 = quantities defined in terms of c_i ($i=1,2$) (equation 12)
l = tubesheet thickness
p = layout pitch (equation 1)
q_{12} = heat flux from region 1 to region 2
r_1 = tube hole radius
r_2 = tube outer radius
r_3 = equivalent ligament radius (Fig. 3)
r', r'' = functions of r_1, r_2, and r_3 (equation 22)
S_i = circumference corresponding to radius r_i ($i=1,2,3$)
T_1 = average cross-sectional temperature of tube wall (function of x)
T_2 = average cross-sectional temperature of tube-sheet ligament (function of x)
T_1' = average tube wall cross-sectional temperature in region 1′
T_t = tubeside fluid temperature
T_s = shellside fluid temperature
α, β = parameters which define tube metal temperature in region 1′
θ = layout angle (Fig. 1)
ω_1, ω_2 = coefficients which relate d_i to b_i (equation 15)

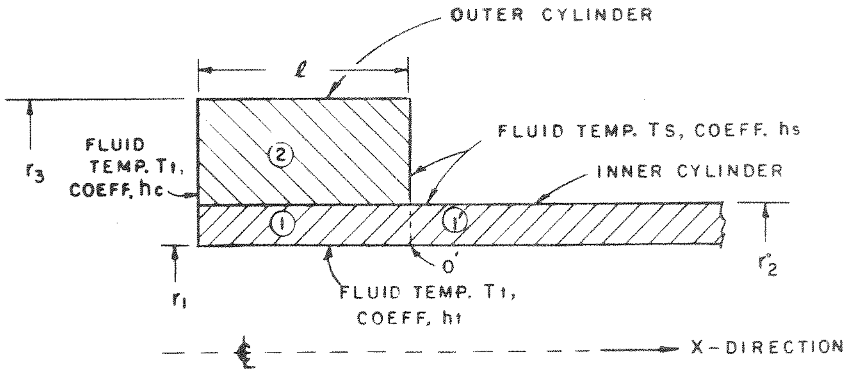

Fig. 2 Cross section of the idealized problem

Table 1 Comparison of analytical solution with finite element method (F.E.M.) results

Location x_1 (inch)	Mean tubewall temp., °F Analytical solution	F.E.M. solution	Fractional difference × 100[a]	Mean ligament temp. °F Analytical solution	F.E.M. solution	Fractional difference × 100[a]
0.0	211.4	210.4	0.50	215.4	213.8	0.80
0.12	215.6	214.4	0.60	218.2	216.8	0.70
0.24	219.7	218.7	0.50	222.1	221.2	0.45
0.37	224.6	224.2	0.20	227.0	226.7	0.15
0.49	230.6	230.9	−.15	233.1	233.6	−.25
0.61	237.7	239.1	−.70	240.4	241.9	−.75
0.73	246.1	248.6	−1.25	249.0	251.6	−1.3
0.85	256.4	260.4	−2.0	258.7	262.8	−2.05
0.98	273.6	268.3	2.65	269.3	268.5	0.40

[a]Fractional difference is defined as (analytical solution - F.E.M. solution) $(T_s - T_t)$

in this paper. Substituting for q_{12} in equations (2) and (4), and rearranging terms, yields

$$\left[k_1 \frac{d^2}{dx^2} - \frac{h_t S_1}{a_1} - \frac{h_{12} S_2}{a_1}\right] T_1 + \frac{h_{12} S_2}{a_1} T_2 + \frac{h_t S_1 T_t}{a_1} = 0 \qquad (6)$$

$$\left[k_2 \frac{d^2}{dx^2} - \frac{h_{12} S_2}{a_2}\right] T_2 + \frac{h_{12} S_2}{a_2} T_1 = 0 \qquad (7)$$

Hence

$$T_1 = \frac{a_2}{h_{12} S_2}\left(\frac{h_{12} S_2}{a_2} - k_2 \frac{d^2}{dx^2}\right) T_2 \qquad (8)$$

Substituting for T_1 in equation (6) yields a fourth order ordinary differential equation in T_2

$$c_1 \frac{d^4 T_2}{dx^4} + c_2 \frac{d^2 T_2}{dx^2} + c_3 T_2 + c_4 = 0 \qquad (9)$$

where

$$c_1 = \frac{-k_1 k_2 a_2}{h_{12} S_2} \qquad (a)$$

$$c_2 = = k_1 + \frac{k_2 a_2}{h_{12} S_2}\left(\frac{h_t S_1}{a_1} + \frac{h_{12} S_2}{a_1}\right) \qquad (b)$$

$$c_3 = \frac{-h_t S_1}{a_1} \qquad (c) \qquad (10)$$

$$c_4 = \frac{h_t S_1 T_t}{a_1} \qquad (d)$$

The complete solution of equation (9) is given in terms of four undetermined constants of integration, b_i $(i = 1, 2, 3, 4)$,

$$T_2 = b_1 e^{m_1 x} + b_2 e^{-m_1 x} + b_3 e^{m_2 x} + b_4 e^{-m_2 x} - \frac{c_4}{c_3} \qquad (11)$$

where

$$m_i = \left\{\frac{-c_2 + (-1)^{i+1}(c_2^2 - 4c_1 c_3)^{1/2}}{2c_1}\right\}^{1/2}; \; i = 1,2 \qquad (12)$$

By virtue of equation (8), we have:

$$T_1 = d_1 e^{m_1 x} + d_2 e^{-m_1 x} + d_3 e^{m_2 x} + d_4 e^{-m_2 x} - \frac{c_4}{c_3} \qquad (13)$$

where

$$d_1 = \omega_1 b_1, \; d_2 = \omega_1 b_2, \; d_3 = \omega_2 b_3, \; d_4 = \omega_2 b_4 \qquad (14)$$

and

$$\omega_i = \left[\frac{-k_2 a_2}{h_{12} s_2} m_i^2 + 1\right]; \quad i = 1,2 \qquad (15)$$

The four constants of integration, b_i, are evaluated by utilizing the boundary conditions at $x = 0, l$. These are

region 2

$$\text{At } x = 0, \; k_2 \frac{dT_2}{dx} = -h_c (T_t - T_2)|x=0 \qquad (a)$$

$$\text{At } x = l, \; k_2 \frac{dT_2}{dx} = -h_s (T_2 - T_s)|x=l \qquad (b)$$

region 1 (16)

$$\text{At } x = 0, \; k_1 \frac{dT_1}{dx} = -h_c (T_t - T_1) \qquad (c)$$

$$\text{At } x = l; \; \frac{dT_1}{dx} = \frac{dT_1'}{dx} |x=0 \text{ in region } 1' \qquad (d)$$

To determine the temperature gradient in region 1′, we model it as a very long tube attached to a source at temperature T^* (say) subject to a heat transfer from a fluid at temperature T_t on its inside surface (coefficient h_t) and from a fluid at temperature T_s on its outside surface (coefficient

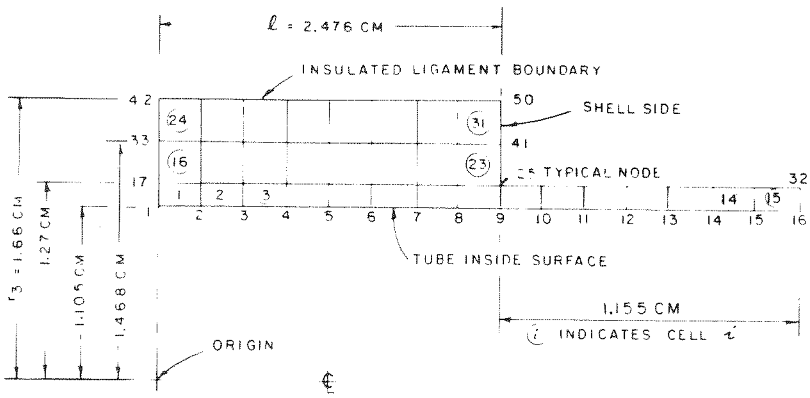

Fig. 3 Finite element grid for the sample problem

h_s'). The solution for the temperature field follows directly from the heat conduction equation

$$T_1' = c_1 \bar{e}^{\alpha x} + c_2 e^{\alpha x} + \frac{\beta^2}{\alpha^2} \tag{17}$$

where

$$\alpha^2 = \frac{h_s' S_2 + h_t S_1}{k_1 a_1} \quad \text{(a)}$$
$$\beta^2 = \frac{h_s' S_2 T_s + h_t S_1 T_t}{k_1 a_1} \quad \text{(b)} \tag{18}$$

region 1′

$x=0$ occurs at point $0'$ in Fig. 2.

We require T_1' to be finite as x becomes large, which implies that $c_2 = 0$. Furthermore, at $x=0$, $T_1' = T_*$* which yields

$$c_1 = T^* - \frac{\beta^2}{\alpha^2}$$

Hence

$$T_1' = \left(T^* - \frac{\beta^2}{\alpha^2}\right)\bar{e}^{\alpha x} + \frac{\beta^2}{\alpha^2}$$

and

$$\frac{dT_1'}{dx}\,|x=0 = -\alpha\left(T^* - \frac{\beta^2}{\alpha^2}\right) \tag{19}$$

equation (16*d*) can be combined with equation (19) to furnish an independent algebraic equation in the undertermined constants, b_i. Equations (16*a*)-(16*c*) provide the other three algebraic equations. The resulting system of four linear algebraic equations can be represented in subscript notation as

$$A_{ij}\, b_j = f_i;\ i=1,4 \tag{20}$$

Summation on the repeated subscript ($j=1, \ldots 4$) is implied in equation (20).

The coefficients of the [A] matrix and the {*f*} vector are given in Appendix B.

Equation (20) is solved for the four b_i's using a standard Gaussian elimination subroutine. The expressions for T_1 and T_2 follow directly thereafter from equations (11) and (13), respectively. In this manner, the temperature field is completely described. At this point, the description for the entire solution scheme is complete, except for the intraregion coefficient h_{12}. We will now devise a simple expression for h_{12}, which is consistent with the level of accuracy of the total solution.

It is well known [4] that the general solution for temperature distribution in an axisymmetric long cylinder is a logarithmic function in the radius, r. Hence, it is reasonable to assume the temperature field to be logarithmic in radius r for the inner and outer cylinders. The unknown constants are evaluated using boundary and interface conditions; thus,

$$t_1(r) = m_1 \ln r + n_1;\ r_1 \le r \le r_2 \quad \text{(a)}$$
$$t_2(r) = m_2 \ln r + n_2;\ r_2 \le r \le r_3 \quad \text{(b)} \tag{21}$$

m_1, m_2, n_1, and n_2 are evaluated under the following conditions:

At $r=r'$, $t=T_1$

$r=r''$; $t=T_2$

$r=r_2$, t is continuous

$$r=r_2,\ k_1\frac{dt}{dr} = k_2\frac{dt}{dr}$$

Let r' and r'' be logarithmic mean radii in regions 1 and 2, respectively, i.e.,

$$r' = \frac{r_2 - r_1}{\ln\dfrac{r_2}{r_1}} \qquad r'' = \frac{r_3 - r_2}{\ln\dfrac{r_2}{r_1}} \tag{22}$$

The foregoing relationships determine the four undetermined coefficients in equation (21).

Recalling that the heat flux at $r = r_2 = h_{12}(T_1 - T_2)$, we obtain the final expression for h_{12}.

$$h_{12} = \frac{-k_1}{r_2 \ln\left[\left(\dfrac{r_2}{r''}\right)^{k_1/k_2}\left(\dfrac{r'}{r_2}\right)\right]} \tag{23}$$

Solution Procedure

A step-by-step description of the solution procedure is given in the following to facilitate direct use of the foregoing analysis without the encumbrance of assimilating the details.

(*a*) Input Data:

(*i*) Geometry Data—tubesheet thickness, l, tube inner and outer radii (r_1 and r_2).

Tube layout pitch, p, layout angle, θ. (Fig. 1)

(*ii*) Thermal property data[1]—tube material conductivity, k_1; tubesheet material conductivity k_2.

k_1; tubesheet material conductivity k_2.

(*iii*) Heat-transfer data—channel side heat-transfer coefficient, h_c; in-tube heat-transfer coefficient, h_t.

Shell side heat-transfer coefficients, h_s, on tubesheet surface; h_s' on tube surface.

[1] If a fouling factor on any of these surfaces is specified, then the corresponding coefficient should be accordingly adjusted to account for fouling.

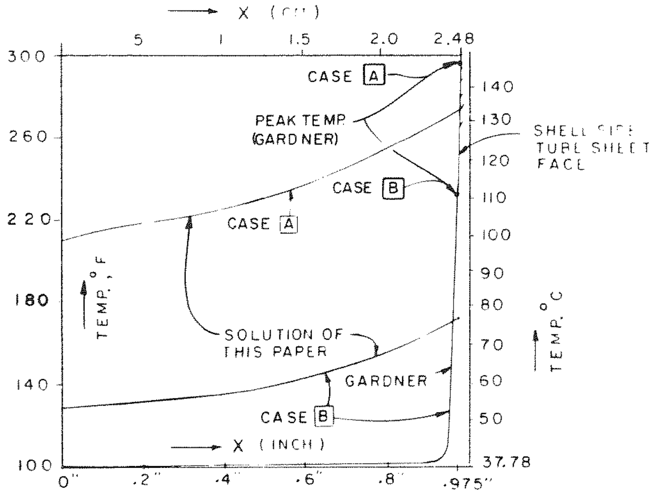

Fig. 4 Tubewall temperature distribution for the example problem

Tubeside fluid temperature, T_t; shell side fluid temperature, T_s.

(*b*) Compute r_3 from equation (1), and h_{12} from equation (22) and (23).

(*c*) Compute S_1, S_2 (equation (3), a_1 and a_2 (equation (3*b*) and (4*b*).

Compute c_1, c_2, c_3 and c_4 from equation (10); and m_i from equation (12); ω_i from equation (15), α and β from equation (18).

(*d*) Compute all 16 coefficients of A_{ij} and four terms of f_i (Appendix B).

(*e*) Solve 4 × 4 linear equation set (equation 20) to obtain values of b_1, b_2, b_3 and b_4.

(*f*) The temperature in the ligament is obtained as a function of x from equation (11). Determine d_i $(i = 1, \ldots 4)$ from equation (14)–(15). The tubewall temperature as a function of x is obtained from equation (13).

Numerical Example

A tubesheet in contact with tubeside fluid at 100°F (37.77°C), and shellside fluid at 300°F (148.89°C) is analyzed.

The basic data is given below:

(*a*) Geometry data—l = 0.975″ (2.48cm); r_1 = 0.435″ (1.105 cm) r_2 = 0.5″ (1.27cm) θ = 60 deg

(*b*) Thermal property data—Tube material conductivity k_1 = 9.3 Btu/Hr.-ft.-°F (16.1 J/msk) Tubesheet material conductivity, k_2 = 30 Btu/hr-ft-°F (51.92 J/msk)

(*c*) Heat-transfer data—channel side coefficient, h_c = 50 Btu/Sq. ft.-°F-Hr. (283.91 J/m²sk) In-tube coefficient, h_t = 50 Btu/Hr.Sq.ft.-°F (283.91 J/m²sk). Shell side coefficients, $h_s = h_s'$ = 1000 Btu/Hr.-Sq ft-°F (5678.3 J/m²sk)

All surface fouling factors are assumed to be zero.

Table 1 shows the average tubewall temperature and average ligament temperature as a function of x. The same problem is solved using a finite element heat conduction-convection computer program. The finite element grid utilized is shown in Fig. 3. The computer program "FWILHEAT" is a general purpose heat-transfer computer program. The temperatures are computed at the node points. From the nodal temperatures, average cross-sectional temperature is computed for a series of values of x. Table 1 summarizes the F.E.M. results alongside the results obtained by the method proposed in this paper. The fractional difference is seen to be less than 3 percent at all locations.

The tubewall temperatures is plotted in Fig. 4 as a function of x_1. Figure 4 also shows the tubewall temperature variation when all coefficients are set equal ($h_t = h_s = h_c$ = 50 Btu/Hr.-Sq.ft.-°F) and the conductivities k_1 and k_2 are also set equal ($k_1 = k_2$ = 9.3 Btu/Hr.-ft.°F). This latter case is labelled as case B. For purposes of comparison, the solution produced by Gardner's formula is also plotted. The discrepancy between the two solutions is obvious. Whereas, Gardner's solutions shows a purely skin effect, the solution given herein indicates a nonlinear thermal gradient through the tubesheet. Therefore, a global thermal stress in the tubesheet will be developed, as opposed to only "peak stress" predicted by the Appendix A formula. Global thermal stresses produced are important in evaluating plastic breakdown of the structure under cyclic conditions. Peak stresses, on the other hand, are only significant in evaluating safety from multiple cycle fatigue. It is also to be noted from the foregoing examples that the skin effect predicted by the Appendix A formula grossly exaggerates the severity of the state of stress on the shellside face of the tubesheet.

Conclusion

A method to determine the temperature field in a typical ligament of a tubesheet has been developed. It is shown, via a numerical example, that significant thermal gradient throughout the body of the tubesheet may exist in ordinary heat exhangers. The thermal stresses so produced are "secondary stresses" rather than "peak stresses." Secondary stresses warrant closer attention in tubesheet design, especially in units subject to large operating cycles. Furthermore, an axial thermal gradient may produce "bowing" of the tubesheet. In designs where the tubesheet also serves as a flange, such a bowing may cause joint leakage. Less perceptibly, bowing of the tubesheet may cause the pass partition plates to lose contact on the tubesheet surface thereby inducing an interpass leakage. Equally important, tubesheet ligament temperature distribution under operating conditions has a great deal of impact on the sealworthiness of "rolled only" joint conditions.

The solution procedure given herein is intended to be used as a routine design tool. A programmable calculator should suffice to furnish a computing ability to handle the task.

Acknowledgment

We are thankful to Mr. Vytas Maciunas for his assistance in computer programing.

References

1 K. A. Gardener, Heat Exchanger Tubesheet Temperatures, The Refiner & Natural Gasoline Manufacturer, Mar. 1942.

2 Standard for Power Plant Heat Exchangers, 1st ed., Heat Exchange Institute, 1980, p. 10.

3 A. I. Soler, "Tube Stresses due to In-Plane Thermal Expansion of Tubesheets in Closely Spaced Double Tubesheets," *International Journal of Pressure Vessels and Piping*, Vol. 7, 1979, pp. 119–132.

4 M. Jakob, *Heat Transfer*, Vol. 1, Wiley & Sons, 1967, p. 133.

APPENDIX A

Gardner's Equations

Setting the origin of the axial coordinate, x, on the shellside face, the temperature of the tubesheet ligament, T, is given by equation (A1).

$$T = T_t + (T_s - T_t)\left[\frac{\text{Cosh}\,\dfrac{\Lambda x}{l} - \dfrac{\Lambda y_2}{y_1}\,\eta\,\text{Sinh}\,\dfrac{\Lambda x}{l}}{1 + \dfrac{\eta h_t}{h_s}}\right] \tag{A1}$$

where x is directed towards the tubeside face, and

$$\Lambda = \left(\frac{y_1}{y_2}\frac{h_t l}{k}\right)^{1/2}$$

$$\eta = \frac{y_1}{y_2\Lambda}\left[\frac{1+\frac{y_1}{y_2\Lambda}\tanh\Lambda}{\frac{y_1}{y_2\Lambda}+\tanh\Lambda}\right] \tag{A2}$$

$$y_1 = 2\pi r_1 \tag{A3}$$

$$y_2 = (p^2 \operatorname{Sin}\theta - \pi r_1{}^2) \tag{A4}$$

In the foregoing, k is the thermal conductivity of tube and tubesheet material (assumed identical); h_t is tubeside heat-transfer coefficient; h_t and h_c are assumed to be equal; h_s and h_s (shellside coefficients) are also assumed to be equal.

APPENDIX B

Coefficients of [A] Matrix and $\{f\}$ Vector

$A_{11} = k_2 m_1 = h_c$; $\quad A_{21} = (k_2 m_1 + h_s)\, e^{m_1 l}$

$A_{12} = k_2 m_1 - h_c$; $\quad A_{22} = (-k_2 m_1 + h_s)\, \bar{e}^{m_1 l}$

$A_{13} = k_2 m_2 - h_c$; $\quad A_{23} = (k_2 m_2 + h_s)\, e^{m_2 l}$

$A_{14} = -k_2 m_2 - h_c$; $\quad A_{24} = (-k_2 m_2 + h_s)\, \bar{e}^{m_2 l}$

$f_1 = -h_c\left(T_t + \frac{c_4}{c_3}\right)$; $\quad f_2 = h_s\left(\frac{c_4}{c_3} + T_s\right)$

$A_{31} = (k_1 m_1 - h_c)\, \omega_1$; $\quad A_{41} = \omega_1(m_1 + \alpha)\, e^{m_1 l}$

$A_{32} = -(k_1 m_1 + h_c)\, \omega_1$; $\quad A_{42} = \omega_1(-m_1 + \alpha)\, \bar{e}^{m_1 l}$

$A_{33} = (k_1 m_2 - h_c)\, \omega_2$ $\quad A_{43} = \omega_2(m_2 + \alpha)\, e^{m_2 l}$

$A_{34} = -(k_1 m_2 + h_c)\, \omega_2$; $\quad A_{44} = \omega_2(-m_2 + \alpha)\, \bar{e}^{m_2 l}$

$f_3 = -h_c\left(T_t + \frac{c_4}{c_3}\right)$; $\quad f_4 = \frac{\beta^2}{\alpha} + \frac{\alpha c_4}{c_3}$